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# Modeling and stability analysis of ecosystem with water saturation constraints

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**ABSTRACT**

The problem of modeling and stability analysis of ecosystem with water saturation constraints is studied in this paper. The signal and system-based analyzing methods are applied to model the ecosystem. By considering the water saturation constraints, a new ecosystem model is established. A sufficient condition for the global asymptotic stability condition of the systems is given by using the linear matrix inequality method. By introducing the parameter matrix into the Lyapunov function, the conservatism of the stability condition is reduced. This method solves well the problems which are usually hard to be solved with traditional method.

*Keywords:* Ecosystem; Water; Saturation constraints; Linear matrix inequality

## 1. Introduction

The concept of ecosystem was put forward by Odum [1] in the 1960s. Numerous public hazards occurred in the world in the 20th century, which deep thinking on the changes of ecological environment caused by human development. Therefore it also triggered an upsurge in the study of ecosystem [2–5].

In the study of ecosystems, even if the specific internal entity is unknown, but from the overall or local effects and stimuli (input signals) and the responses (responses) produced by their results, a system model can be built with data processing of input signals [6,7]. Through empirical or regressive analysis, each component is quantified and the general situation is obtained. But ecosystem is a complex large-scale system. For the convenience of research, some of these factors can be analyzed separately, such as illumination, temperature, moisture, mineral elements, introduction or removal of a species, felling and grazing, etc. At present, some papers have studied the compartment model of ecological engineering and applied it to practical engineering design [8,9]. But in the quantitative calculation of ecological quantitative process and predicting the future state of ecosystem, more precise systematic research methods are often needed.

In addition, saturation phenomenon exists widely in various systems. If the saturation constraint is not considered, then the performance of the systems will be reduced, or the systems will be even unstable under severe conditions [10–12]. Then Schindler et al. [13] introduced the design method of saturation system into the saturated network control systems, which studied the output feedback stabilization of the saturated network systems. Then, some scholars studied the time-delay systems with saturation constraints. Samsudin et al. [14] have studied the time-delay switching systems with nonlinear actuator saturation, and designed the state feedback controller with the linear matrix inequality method. Sitch et al. [15] have studied the global asymptotic stability of multiple-input multiple-output saturated time-delay systems, and estimated the attraction region by iterative algorithm. In the actual engineering control process, the control input often needs to meet certain conditions, and the actuator saturation is the most common control constraint. Therefore, the research on the actuator saturation control is of great practical significance [16–18].

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However, in the research process of modeling and stability analysis of ecosystem, the above reference lacks the water saturation constraints. Because of this, based on the predecessors' research, by considering the influence of water saturation, this paper introduces signal and system theory into ecological modeling, establishes a more realistic system model, and then obtains asymptotic stability sufficient conditions of the ecosystem [19–24].

## 2. Preliminaries

In fact, the ecological processes and interactions among the various components of an ecosystem are a manifestation of control and feedback control. Deforestation of trees, migration of animals, and hunting all inevitably lead to the loss of nitrogen [25,26]. Let $f(t)$ denote the rate of microbial uptake of nitrogen. Nitrogen contents in forests, microorganisms and animals are expressed as state variables $x_1(t)$, $x_2(t)$, $x_3(t)$, respectively. According to the change of nitrogen element in the system and the relationship between input and output, the state equation describing the system can be written as follows:

$$\begin{aligned}\dot{x}_1(t) &= -c_1x_1(t)+c_2x_2(t)-c_3x_1(t)-c_4x_1(t)\\ \dot{x}_2(t) &= -c_3x_1(t)+c_2x_2(t)-c_5x_1(t)+f(t)\\ \dot{x}_3(t) &= c_4x_1(t)-c_5x_2(t)-c_6x_1(t)\end{aligned} \tag{1}$$

The matrix form of the state equation is as follows:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}-c_1-c_3-c_4 & c_2 & 0\\ -c_3 & c_2 & -c_5\\ c_4 & 0 & -c_5-c_6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}f(t) \tag{2}$$

Considering the saturation of water resources in ecosystem, the saturation term is added to the system model. In addition, considering the time delay in the process directly affected by each component of the systems, time delay is added to the system. By considering multiple factors, the ecosystem model can be modeled as:

$$\begin{aligned}\dot{x}(t) &= Ax(t)+A_dx(t-d)+B\text{sat}(u(t))\\ x(t) &= \phi(t) \qquad t\in[-d,\ 0]\end{aligned} \tag{3}$$

where $x(t) \in R^n$ is state vector, $u(t) \in R^m$ is control input vector, $A, A_d \in R^{n\times n}$ are systems matrices, $B \in R^{n\times m}$ is control input matrix, $\varphi(t) = [\varphi_1(t) \ \ \varphi_2(t) \ \ \cdots \ \ \varphi_n(t)]^T \in R^n$ is the given initial state of the systems. $d$ is state delay, the saturation function is as follows:

$$\text{sat}(u(t)) = \left[\text{sat}(u_1(t)),\text{sat}(u_2(t)),\cdots,\text{sat}(u_m(t))\right] \tag{4}$$

and

$$\text{sat}(u_i(t)) = \begin{cases}\underline{u}_i & u_i(t)\le \underline{u}_i<0\\ u_i(t) & \underline{u}_i\le u_i(t)\le \bar{u}_i\\ \bar{u}_i & 0<\bar{u}_i\le u_i(t)\end{cases} \tag{5}$$

The state feedback controller of the systems (Eq. (3)) will be designed such as:

$$u(t) = 2Kx(t) \tag{6}$$

where $K\in R^{m\times n}$ is a constant matrix. Substituting Eq. (6) into the system Eq. (3) to get the closed-loop system:

$$\begin{aligned}\dot{x}(t) &= \bar{A}x(t)+A_dx(t-d)+B\eta(t)\\ x(t) &= \phi(t) \qquad t\in[-d,\ 0]\end{aligned} \tag{7}$$

where $\bar{A} = A + BK$, $\eta(t) = \text{sat}(2Kx(t)) - Kx(t)$, and satisfying:

$$\eta^T(t)\eta(t) \le x^T(t)K^TKx(t) \tag{8}$$

The purpose of this paper is to determine the controller such as (Eq. (6)), so that the closed-loop system Eq. (7) is asymptotically stable.

*Lemma 1* [8]: For a given symmetric matrix of $n$ order $S = \begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}$, where $S_{11}$ is the $r$ order, the following three conditions are equivalent

(i) $S < 0$

(ii) $S_{11} < 0$, $S_{22} - S_{11}^TS_{11}^{-1}S_{12} < 0$

(iii) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$

## 3. Main results

*Theorem 1*: If there is constant $\varepsilon > 0$, symmetric positive definite matrices $P, Q \in R^{n\times n}$ and matrices $K \in R^{m\times n}$ make the following matrix inequalities hold

$$\Theta = \begin{bmatrix}\bar{A}^TP+P\bar{A}+Q+\varepsilon K^TK & PA_d & PB\\ * & -Q & 0\\ * & * & -\varepsilon I\end{bmatrix} < 0 \tag{9}$$

the closed-loop system Eq. (7) is asymptotically stable.

*Proof*: Selecting the Lyapunov function

$$V(t) = x^T(t)Px(t)+\int_{t-d}^{t}x^T(s)Qx(s)\,ds \tag{10}$$

where $P, Q \in R^{n\times n}$ are undetermined symmetric positive definite matrices.

Along the system Eq. (7), get the derivative of the function $V(t)$

$$\begin{aligned}\dot{V}(t) &= 2x^T(t)P\dot{x}(t)+x^T(t)Qx(t)-x^T(t-d)Qx(t-d)\\ &= x^T(t)(P\bar{A}+\bar{A}^TP)x(t)+2x^T(t)PA_dx(t-d)+2x^T(t)PB\eta(t)\\ &\quad +x^T(t)Qx(t)-x^T(t-d)Qx(t-d)\\ &= \Phi^T(t)\begin{bmatrix}P\bar{A}+\bar{A}^TP & PA_d & PB\\ * & -Q & 0\\ * & * & 0\end{bmatrix}\Phi(t)\end{aligned} \tag{11}$$

where

$$\Phi(t)=\begin{bmatrix} x(t) \\ x(t-h) \\ \eta(t) \end{bmatrix}$$

From the inequality Eq. (4), obtain

$$0 \le \Phi^T(t)\begin{bmatrix} \varepsilon K^T K & 0 & 0 \\ * & 0 & 0 \\ * & * & -\varepsilon I \end{bmatrix}\Phi(t) \tag{12}$$

where ε is an arbitrary small positive number.

Inserting the above inequality into Eq. (6), we obtain

$$\dot{V}(t) \le \Phi^T(t)\Theta\Phi(t) \tag{13}$$

where

$$\Theta=\begin{bmatrix} \bar{A}^T P + P\bar{A} + Q + \varepsilon K^T K & PA_d & PB \\ * & -Q & 0 \\ * & * & -\varepsilon I \end{bmatrix}$$

With the Lyapunov stability theory, it is known that when the condition Eq. (5) is established, the closed-loop system Eq. (7) is asymptotically stable.

*Theorem 2*: If there is constant $\varepsilon > 0$, symmetric positive definite matrices $X$, $\bar{Q} \in R^{n\times n}$ and matrices $\bar{K} \in R^{m\times n}$ make the following linear matrix inequalities hold

$$\begin{bmatrix} AX + B\bar{K} + X^T A^T + \bar{K}^T B^T + \bar{Q} & A_d X & \bar{\varepsilon} B & \bar{K}^T \\ * & -\bar{Q} & 0 & 0 \\ * & * & -\bar{\varepsilon} I & 0 \\ * & * & * & -\bar{\varepsilon} I \end{bmatrix} < 0 \tag{14}$$

The closed-loop system Eq. (7) is asymptotically stable by selecting the state feedback controller $u(t) = 2\bar{K}X^{-1}x(t)$.

*Proof*: With the lemma 1, we know that inequality Eq. (5) is equivalent to

$$\begin{bmatrix} P\bar{A} + \bar{A}^T P + Q & PA_d & PB & \varepsilon K^T \\ * & -Q & 0 & 0 \\ * & * & -\varepsilon I & 0 \\ * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{15}$$

The matrix $\operatorname{diag}\{P^{-1}, P^{-1}, \varepsilon^{-1} I, \varepsilon^{-1} I\}$ is multiplied at both sides of the above inequality, then we obtain

$$\begin{bmatrix} \bar{A}P^{-1} + P^{-1}\bar{A}^T + P^{-1}QP^{-1} & A_d P^{-1} & \varepsilon^{-1} B & P^{-1}K^T \\ * & -P^{-1}QP^{-1} & 0 & 0 \\ * & * & -\varepsilon^{-1} I & 0 \\ * & * & * & -\varepsilon^{-1} I \end{bmatrix} < 0 \tag{16}$$

Inserting $\bar{A} = A + BK$ into the above inequality

$$\begin{bmatrix} (A+BK)P^{-1} + P^{-1}(A+BK)^T + P^{-1}QP^{-1} & A_d P^{-1} & \varepsilon^{-1} B & P^{-1}K^T \\ * & -P^{-1}QP^{-1} & 0 & 0 \\ * & * & -\varepsilon^{-1} I & 0 \\ * & * & * & -\varepsilon^{-1} I \end{bmatrix} < 0 \tag{17}$$

And make some substitutions such as $X = P^{-1}$, $\bar{Q} = P^{-1}QP^{-1}$, $\bar{K} = KP^{-1}$, $\bar{\varepsilon} = \varepsilon^{-1}$, the above inequality and Eq. (14) are equivalent.

## 4. Conclusion

In this paper, for a class of ecosystem with water saturation constraints, by using the linear matrix inequality method, the asymptotic stability condition is given. By introducing the parameter matrix in the Lyapunov function, the conservatism of the system stability condition is reduced. Compared with traditional methods, the method in this paper is more practical and effective.

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